

Symplectic non-squeezing in Hilbert space and discrete Schrödinger equations

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Abstract. We prove a generalization of Gromov's symplectic non-squeezing theorem for the case of Hilbert spaces. Our approach is based on filling almost complex Hilbert spaces by complex discs partially extending Gromov's results on existence of J -complex curves. We apply our result to the flow of the discrete nonlinear Schrödinger equation.

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1 Introduction

In the space \mathbb{R}^{2n} with coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ and standard symplectic form $\omega = \sum_j dx_j \wedge dy_j$, we consider the Euclidean unit ball \mathbb{B} and the cylinder $\Sigma = \{(x, y) : x_1^2 + y_1^2 < 1\}$. Gromov's non-squeezing theorem [14] states that if for some $r, R > 0$ there exists a symplectic embedding $f : r\mathbb{B} \rightarrow R\Sigma$, that is, $f^*\omega = \omega$, then $r \leq R$. This result had a deep impact on the development of the symplectic geometry. In contrast to the case of finite-dimensional symplectic manifolds arising from the classical mechanics and dynamics, the symplectic structures and flows corresponding Hamiltonian PDEs are defined on suitable Hilbert spaces, usually Sobolev spaces (see for instance [17]). This explains the interest in analogs of Gromov's theorem for symplectic Hilbert spaces. The first non-squeezing result for symplectic flows of various classes of Hamiltonian PDEs was obtained by Kuksin [16] and later extended in the work of Bourgain [6, 7], Colliander, Keel, Staffilani, Takaoka, and Tao [10], Roumégoux [19]. Their approach is based on approximation of a symplectic flow on a Hilbert space by finite-dimensional symplectic flows which reduces the situation to Gromov's theorem. It seems natural to look for a general analog of Gromov's theorem for symplectic Hilbert spaces. Abbondandolo and Majer [1] prove the result in the case where the symplectic image $f(r\mathbb{B})$ of the Hilbert ball $r\mathbb{B}$ is convex. Finally, Fabert [12] has recently proposed a proof of the result for general symplectic flows in Hilbert spaces using non-standard analysis.

In the present work we prove a generalization of Gromov's non-squeezing theorem to the case of symplectic Hilbert spaces under assumptions of boundedness and regularity of the symplectic transformation in certain Hilbert scales. Gromov's original proof uses almost complex structures J tamed by the standard symplectic form on the complex projective space \mathbb{CP}^n ; the key technical tool is filling the projective space by J -complex spheres. An immediate attempt to extend this construction to the case of Hilbert spaces leads to difficulties because the main ingredients of Gromov's theory (compactness and transversality for J -complex curves) are not available. We use the method introduced in our previous paper [21] where we give a new simple proof of Gromov's theorem. This approach can be extended with suitable modifications to the Hilbert space case. The main idea is to replace J -complex spheres in Gromov's argument by J -complex discs with boundaries attached to the boundary of a cylinder. These discs are Hilbert space valued functions satisfying a certain first order quasilinear system of PDE with non-linear boundary conditions. The integral equation corresponding to this boundary value problem has a solution by combination of the contraction mapping principle and the Schauder fixed point theorem. Similar methods are known in the theory of the scalar Beltrami equation which partially inspired our approach; we extend them to vector valued functions. In the last section, we apply our main result to the flows of infinite systems of ODEs, in particular, discrete nonlinear Schrödinger equations.

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2 Almost complex structures on Hilbert spaces

In this section we introduce almost complex structures (see [5]), spaces of vector-valued functions, and Hilbert scales (see [17]). We include some auxiliary results concerning almost complex structures in Hilbert spaces because we could not find precise references.

2.1 Almost complex and symplectic structures

Let V be a real vector space. If V has finite dimension, then we assume that $\dim V$ is even. A *linear almost complex structure* J on V is a bounded linear operator $J : V \rightarrow V$ satisfying $J^2 = -I$. Here and below I denotes the identity map or the identity matrix depending on the context.

Let \mathbb{H} be a complex Hilbert space with Hermitian scalar product $\langle \bullet, \bullet \rangle$; we consider only *separable* Hilbert spaces. Fix an orthonormal basis $\{e_j\}_{j=1}^\infty$ of \mathbb{H} such that $Z = \sum_{j=1}^\infty Z_j e_j$ for every $Z \in \mathbb{H}$. Here $Z_j = x_j + iy_j = \langle Z, e_j \rangle$ are complex coordinates of Z . Then \mathbb{H} can be identified with the complex space l^2 . We will use complex conjugation $\bar{Z} = \sum_{j=1}^\infty \bar{Z}_j e_j$. The *standard almost complex structure* J_{st} on \mathbb{H} is the operator defined as $J_{st}Z = iZ$, hence, $J_{st}^2 = -I$. In the case where \mathbb{H} has a finite dimension n the structure J_{st} is the usual complex structure on \mathbb{C}^n . We do not specify the dimension (finite or infinite) in this notation since it will be clear from the context. Denote by $\mathcal{L}(\mathbb{H})$ the space of real linear bounded operators on \mathbb{H} . An *almost complex structure* J on \mathbb{H} is a continuous map $J : \mathbb{H} \rightarrow \mathcal{L}(\mathbb{H})$, $J : \mathbb{H} \ni Z \rightarrow J(Z)$ such that every $J(Z)$ satisfies $J^2(Z) = -I$. If the map $Z \mapsto J(Z)$ is independent of Z , then we can identify the tangent space of \mathbb{H} at Z with \mathbb{H} and view J as a linear almost complex structure on \mathbb{H} .

Denote by $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ the unit disc in \mathbb{C} . It is equipped with the standard complex structure J_{st} of \mathbb{C} . Let J be an almost complex structure on \mathbb{H} . A C^1 -map $f : \mathbb{D} \rightarrow \mathbb{H}$ is called a *J-complex disc* in \mathbb{H} if it satisfies the *Cauchy-Riemann equations*

$$J \circ df = df \circ J_{st}. \quad (1)$$

We use the complex derivatives

$$f_Z = \frac{\partial f}{\partial Z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad f_{\bar{Z}} = \frac{\partial f}{\partial \bar{Z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right),$$

where f is a map between two (finite or infinite dimensional) Hilbert spaces. It is convenient to rewrite (1) in complex notation. Assume that for all $Z \in \mathbb{H}$ the operator $(J_{st} + J)(Z)$ is invertible. Then the linear operator

$$L := (J_{st} + J)^{-1}(J_{st} - J)$$

is well defined. Like in the finite-dimensional case (see [5]), the operator L is J_{st} -anti-linear, i.e., $J_{st}L = -LJ_{st}$. Hence, there exists a bounded J_{st} -linear operator $A_J : \mathbb{H} \rightarrow \mathbb{H}$ such that

$$Lh = A_J \bar{h}.$$

We call A_J the *complex representation* of J and often omit J . With this convention the Cauchy-Riemann equations (1) for a J -complex disc $Z : \mathbb{D} \rightarrow \mathbb{H}$, $Z : \mathbb{D} \ni \zeta \mapsto Z(\zeta)$ can be written in the form

$$Z_{\bar{\zeta}} = A_J(Z) \bar{Z}_{\bar{\zeta}}, \quad \zeta \in \mathbb{D}. \quad (2)$$

The *standard symplectic form* ω on \mathbb{H} is a nondegenerate antisymmetric bilinear form defined by

$$\omega = \frac{i}{2} \sum_{j=1}^{\infty} dZ_j \wedge d\bar{Z}_j.$$

We use the natural identification of \mathbb{H} with its tangent space at every point.

For a map $Z : \mathbb{D} \rightarrow \mathbb{H}$, $Z : \zeta \mapsto Z(\zeta)$ its (symplectic) *area* is defined by

$$\text{Area}(Z) = \int_{\mathbb{D}} Z^* \omega \quad (3)$$

similarly to the finite-dimensional case. If Z is J_{st} -holomorphic, then (3) represents its area induced by the inner product of \mathbb{H} .

2.2 Symplectomorphisms

Let \mathbb{H} be a complex Hilbert space with fixed basis and the standard symplectic form ω . By default, all linear operators are bounded. For an \mathbb{R} -linear operator $F : \mathbb{H} \rightarrow \mathbb{H}$ we denote by F^* its adjoint, that is $\text{Re} \langle Fu, v \rangle = \text{Re} \langle u, F^*v \rangle$. Put

$$\bar{F}Z = \overline{(F\bar{Z})}, \quad \text{and} \quad F^t = \bar{F}^*.$$

Thus F^t is the transpose of F .

Every \mathbb{R} -linear operator $F : \mathbb{H} \rightarrow \mathbb{H}$ can be uniquely written in the form

$$Fu = Pu + Q\bar{u},$$

where P and Q are \mathbb{C} -linear operators. For brevity we write

$$F = \{P, Q\}.$$

Note that

$$F^* = \{P^*, Q^t\}, \quad F^t = \{P^t, Q^*\}.$$

The following two lemmas are proved in [22].

Lemma 2.1 *Let $F = \{P, Q\}$. Then F preserves ω , i.e., $\omega(Fu, Fv) = \omega(u, v)$ if and only if*

$$P^*P - Q^t\overline{Q} = I \text{ and } P^t\overline{Q} - \overline{Q}^tP = 0. \quad (4)$$

A linear operator $F : \mathbb{H} \rightarrow \mathbb{H}$ is called a *linear symplectomorphism* if F is invertible and preserves ω .

Lemma 2.2 *Let $F = \{P, Q\}$ be a linear symplectomorphism. Then F^t also preserves ω , that is,*

$$PP^* - QQ^* = I \text{ and } PQ^t - QP^t = 0. \quad (5)$$

Proposition 2.3 *Let $F = \{P, Q\}$ be a linear symplectomorphism. Then*

- (a) $F^{-1} = \{P^*, -Q^t\}$;
- (b) P is invertible;
- (c) $\|Q\overline{P}^{-1}\| = \|Q\|(1 + \|Q\|^2)^{-1/2} < 1$.

Proof. For convenience, we include the proof from [22]. Part (a) follows by (4) and (5). By (4) and (5), spectral values of the self-adjoint operators PP^* and P^*P are not smaller than 1. Then both P^*P and PP^* are invertible which gives (b). For (c), put $A = Q\overline{P}^{-1}$. We estimate $\|A\| = \|AA^*\|^{1/2}$. By (4) and (5) respectively, we have $Q\overline{P}^{-1} = \overline{P^t}^{-1}Q^t$ and $Q^t(P^t)^{-1} = P^{-1}Q$. Using the latter, $AA^* = (PP^*)^{-1}QQ^*$. Since $PP^* = I + QQ^*$ and QQ^* is self-adjoint, by the spectral mapping theorem

$$\|AA^*\| = \frac{\|QQ^*\|}{1 + \|QQ^*\|} = \frac{\|Q\|^2}{1 + \|Q\|^2}$$

because the function $\lambda \mapsto \lambda(1 + \lambda)^{-1}$ is increasing for $\lambda > 0$. ■

A C^1 -diffeomorphism (continuously Fréchet differentiable map) $\Phi : \Omega_1 \rightarrow \Omega_2$ between two open subsets Ω_j in (\mathbb{H}, ω) is called a *symplectomorphism* if $\Phi^*\omega = \omega$. Here the star denotes the pull-back. In the proof of one of our main results (Theorem 3.1), we encounter an almost complex structure J arising as the direct image

$$J = \Phi_*(J_{st}) := d\Phi \circ J_{st} \circ d\Phi^{-1}$$

of J_{st} under a symplectomorphism $\Phi : (\mathbb{H}, \omega) \rightarrow (\mathbb{H}, \omega)$. We claim that such almost complex structure J has a complex representation A_J , so the Cauchy-Riemann equations for J can be written in the form (2).

Lemma 2.4 *Let $\Phi : \Omega_1 \rightarrow \Omega_2$ be a diffeomorphism of class C^1 between two open subsets Ω_j , $j = 1, 2$ of \mathbb{H} . Put $P = \Phi_Z$ and $Q = \Phi_{\overline{Z}}$. Then the complex representation A_J of the direct image $J = \Phi_*(J_{st})$ has the form*

$$A_J = Q\overline{P}^{-1} \quad (6)$$

Indeed, by Proposition 2.3 the operator P is invertible for all Z . Then the conclusion follows by Lemma 2.3 from [20], whose proof goes through for the Hilbert space case without changes.

2.3 Hilbert scales

Let \mathbb{H} be a complex Hilbert space with fixed basis. Let $(\theta_n)_{n=1}^\infty$ be a sequence of positive numbers such that $\theta_n \rightarrow \infty$ as $n \rightarrow \infty$, for example, $\theta_n = n$. Introduce a diagonal operator

$$D = \text{Diag}(\theta_1, \theta_2, \dots).$$

For $s \in \mathbb{R}$ we define \mathbb{H}_s as a Hilbert space with the following inner product and norm:

$$\langle x, y \rangle_s = \langle D^s x, D^s y \rangle, \quad \|x\|_s = \|D^s x\|.$$

Thus $\mathbb{H}_0 = \mathbb{H}$, $\mathbb{H}_s = \{x \in \mathbb{H} : \|x\|_s < \infty\}$ for $s > 0$, and \mathbb{H}_s is the completion of \mathbb{H} in the above norm for $s < 0$. The family (\mathbb{H}_s) is called a Hilbert scale corresponding to the sequence (θ_n) . For $s > r$, the space \mathbb{H}_s is dense in \mathbb{H}_r , and the inclusion $\mathbb{H}_s \subset \mathbb{H}_r$ is compact. We refer to [17] for details.

We need a version of Proposition 2.3 for Hilbert scales.

Proposition 2.5 *Let (\mathbb{H}_s) be a Hilbert scale. Let $F = \{P, Q\}$ be a linear symplectomorphism of the standard symplectic structure on $\mathbb{H} = \mathbb{H}_0$. Let $s_0, C > 0$ be constants such that $\|F\|_s \leq C$ and $\|F^{-1}\|_s \leq C$ for $0 \leq s \leq s_0$. Then there exist constants $s_1 > 0$ and $0 < a < 1$ depending only on s_0 and C such that for $0 \leq s \leq s_1$*

$$(a) \quad \|P^{-1}\|_s \leq 2C;$$

$$(b) \quad \|Q\bar{P}^{-1}\|_s \leq a.$$

Lemma 2.6 *Let Q be a linear operator in \mathbb{H}_0 . Suppose $\|Q\|_s \leq C$ for real $|s| \leq s_0$. Then $\|D^s Q D^{-s} - Q\|_0 \leq 2Cs_0^{-1}|s|$ for $|s| \leq s_0$.*

Proof. One easily verifies $\|Q\|_s = \|D^s Q D^{-s}\|_0$. Introduce $f(s) = D^s Q D^{-s} - Q$ as a holomorphic function of complex variable s . Since for $t \in \mathbb{R}$ the operator D^{it} is unitary, we have $\|f(s)\|_0 \leq 2C$ in the strip $|\text{Re } s| \leq s_0$, in particular, in the disc $|s| \leq s_0$. Since $f(0) = 0$, by the Schwarz lemma we get the desired estimate. ■

Proof of Proposition 2.5. Since $Pz = \frac{1}{2}(Fz - iF(iz))$ and $Qz = \frac{1}{2}(F\bar{z} + iF(i\bar{z}))$, we have $\|P\|_s \leq C$ and $\|Q\|_s \leq C$. By Proposition 2.3 (a) we also have $\|P^*\|_s \leq C$ and $\|Q^*\|_s \leq C$, here the stars stand for the adjoints in \mathbb{H}_0 . By (5), $P^{-1} = P^*(I + QQ^*)^{-1}$, which we will use to estimate P^{-1} .

Since $\|Q\|_s \leq C$ for $0 \leq s \leq s_0$, we have $\|D^s Q D^{-s}\|_0 \leq C$ for such s . Passing to the adjoint, $\|D^{-s} Q^* D^s\|_0 \leq C$ for $0 \leq s \leq s_0$. Since $\|Q^*\|_s \leq C$ also, we have $\|D^s Q^* D^{-s}\|_0 \leq C$ for $0 \leq s \leq s_0$. Altogether, $\|D^s Q^* D^{-s}\|_0 \leq C$ for all $|s| \leq s_0$. By Lemma 2.6, $\|D^s Q^* D^{-s} - Q^*\|_0 \leq 2Cs_0^{-1}|s|$ for $|s| \leq s_0$.

Denote by Q_s^* the adjoint of Q in \mathbb{H}_s . One easily verifies $Q_s^* = D^{-2s} Q^* D^{2s}$. We rewrite $(I + QQ^*)^{-1} = (I + K + L)^{-1}$, here $K = QQ_s^* \geq 0$ in \mathbb{H}_s and $L = Q(Q^* - Q_s^*)$. We claim that L is small for small s . Indeed,

$$\|Q^* - Q_s^*\|_s = \|D^s(Q^* - D^{-2s} Q^* D^{2s})D^{-s}\|_0 \leq \|D^s Q^* D^{-s} - Q^*\|_0 + \|D^{-s} Q^* D^s - Q^*\|_0 \leq 4Cs_0^{-1}s$$

for $0 \leq s \leq s_0$. Then $\|L\|_s \leq 4C^2 s_0^{-1} s$ for such s . Put $s_1 = \frac{1}{8}C^{-2}s_0$. Then $\|L\|_s \leq \frac{1}{2}$ for $0 \leq s \leq s_1$.

Since $K \geq 0$ in \mathbb{H}_s , we have $\|(I + K)^{-1}\|_s \leq 1$, moreover,

$$\|(I + QQ^*)^{-1}\|_s = \|(I + (I + K)^{-1}L)^{-1}(I + K)^{-1}\|_s \leq 2.$$

Hence $\|P^{-1}\|_s \leq 2C$ for $0 \leq s \leq s_1$ as stated in (a).

We now estimate $A = Q\overline{P}^{-1}$. By Proposition 2.3, $\|A\|_0 \leq a_0 := C(1 + C^2)^{1/2} < 1$. By part (a), $\|A\|_s \leq a_1 := 2C^2$, $0 \leq s \leq s_1$. By interpolation (see [17]) for $0 \leq s \leq s_1$,

$$\|A\|_s \leq a_0^{1-(s/s_1)} a_1^{s/s_1} \rightarrow a_0 \quad \text{as } s \rightarrow 0.$$

Let $a_0 < a < 1$, say $a = \frac{1+a_0}{2}$. Now by shrinking s_1 if necessary we obtain $\|A\|_s \leq a$ for all $0 \leq s \leq s_1$, as desired. ■

2.4 Vector-valued Sobolev spaces

Let X be a Banach space. Denote by $W^{k,p}(\mathbb{D}, X)$ the Sobolev classes of maps $Z : \mathbb{D} \rightarrow X$ admitting weak partial derivatives $D^\alpha Z \in L^p(\mathbb{D}, X)$ up to the order k (as usual we identify functions coinciding almost everywhere). We define weak derivatives in the usual way using the space $C_0^\infty(\mathbb{D})$ of smooth scalar-valued test functions with compact support in \mathbb{D} . Then $W^{k,p}(\mathbb{D}, X) = L^p(\mathbb{D}, X)$ if $k = 0$. The norm on $L^p(\mathbb{D}, X)$ is defined by

$$\|Z\|_{L^p(\mathbb{D}, X)} = \left(\int_{\mathbb{D}} \|Z(\zeta)\|_X^p d^2\zeta \right)^{1/p}.$$

Here $d^2\zeta := (i/2)d\zeta \wedge d\bar{\zeta}$ denotes the standard Lebesgue measure in \mathbb{R}^2 . The space $W^{k,p}(\mathbb{D}, X)$ equipped with the norm

$$\|Z\| = \left(\sum_{|\alpha| \leq k} \|D^\alpha Z\|_{L^p(\mathbb{D}, X)}^p \right)^{1/p}$$

is a Banach space. We use the standard notation $C^\alpha(\mathbb{D}, X)$ for the Lipschitz space. Denote also by $C(\overline{\mathbb{D}}, X)$ the space of vector functions continuous on $\overline{\mathbb{D}}$ equipped with the sup-norm. We will deal with the case $X = \mathbb{H}_s$. We will need the following analog of Sobolev's compactness theorem.

Proposition 2.7 *The inclusion*

$$W^{1,p}(\mathbb{D}, \mathbb{H}_r) \subset C(\overline{\mathbb{D}}, \mathbb{H}_s), \quad s < r, \quad p > 2 \tag{7}$$

is compact.

This result is well-known [4] in the case of vector functions on an interval of \mathbb{R} .

Proof. We decompose (7) into

$$W^{1,p}(\mathbb{D}, \mathbb{H}_r) \subset C^\alpha(\mathbb{D}, \mathbb{H}_r) \subset C(\overline{\mathbb{D}}, \mathbb{H}_s). \quad (8)$$

The first inclusion in (8) is Morrey's embedding with $\alpha = (p - 2)/p$ (see, for example, [22]). The second inclusion is compact by the Arzela-Ascoli theorem, hence (7) is compact. ■

Sobolev's compactness from Proposition 2.7 plays an important role in our argument. It replaces in some sense Gromov's compactness for pseudo-holomorphic curves. This is the main reason we use Hilbert scales. We finally note that the system (2) still makes sense for $Z \in W^{1,p}(\mathbb{D})$ with $p > 2$.

3 Main results

Let \mathbb{H} be a complex Hilbert space with fixed orthonormal basis and the standard symplectic form ω . Let (\mathbb{H}_s) be a Hilbert scale, so that $\mathbb{H}_0 = \mathbb{H}$. Denote by

$$\mathbb{B}^\infty = \{Z \in \mathbb{H} : \|Z\| < 1\}$$

the unit ball in \mathbb{H}_0 . Then $r\mathbb{B}^\infty$ is the ball of radius $r > 0$. We now use the notation

$$Z = (z, w) = (z, w_1, w_2, \dots)$$

for the coordinates in \mathbb{H}_0 . Here $z = \langle Z, e_1 \rangle \in \mathbb{C}$. For a domain $\Omega \subset \mathbb{C}$ we define the cylinder

$$\Sigma_\Omega = \{Z \in \mathbb{H}_0 : z \in \Omega\}$$

in \mathbb{H}_0 . Our first main result is the following

Theorem 3.1 (*Non-squeezing theorem.*) *Let $r, R > 0$ and let G be an open subset in $\Sigma_{R\mathbb{D}}$. Suppose that there exists a symplectomorphism $\Phi : r\mathbb{B}^\infty \rightarrow G$ of class C^1 with respect to the \mathbb{H}_0 -norm. Let $s_0 > 0$. Suppose that for every $s \in [0, s_0]$ the tangent maps $d\Phi(Z) : \mathbb{H}_0 \rightarrow \mathbb{H}_0$ as well as their inverses $(d\Phi(Z))^{-1}$ are in fact bounded in the \mathbb{H}_s norm (as operators $\mathbb{H}_s \rightarrow \mathbb{H}_s$) uniformly in $Z \in r\mathbb{B}^\infty$. Then $r \leq R$.*

Obviously, the scale regularity assumption holds in the finite-dimensional case. Hence, Theorem 3.1 generalizes Gromov's theorem.

In view of interpolation theorems for linear operators in Hilbert scales (see, for example, [17]), it suffices to assume that the tangent maps for Φ are uniformly bounded for $s = 0$ and $s = s_0$.

We also note that by Proposition 2.3 the boundedness of the inverses $(d\Phi(Z))^{-1}$ automatically follows from the boundedness of $d\Phi(Z)$ for $s = 0$, but not for $s > 0$. Instead of the assumption on the inverses, we can assume that the tangent maps $d\Phi(Z)$ are uniformly bounded in \mathbb{H}_s for $|s| \leq s_0$, that is, also for negative s .

It may seem reasonable to require the boundedness of $d\Phi(Z)$ only for $Z \in \mathbb{H}_s$. However, by the principle of uniform boundedness and continuity of $d\Phi$, it would imply uniform boundedness on the whole ball, hence our hypothesis does not restrict generality.

This is clear that Theorem 3.1 can be applied to symplectomorphisms between an arbitrary ball (not necessarily centered at the origin) and a cylinder obtained from $\Sigma_{R\mathbb{D}}$ by an affine translation and a permutation of coordinates because such transformations are symplectic.

Theorem 3.1 is a consequence of our second main result on the existence of J -complex discs. Following [21], we replace a circular cylinder by a triangular one. The reason is that the construction of J -complex discs in a circular cylinder leads to a boundary value problem for the Cauchy-Riemann equations with non-linear boundary conditions. For the triangular cylinder, the boundary conditions become linear although with discontinuous coefficients. The latter can be handled by means of modified Cauchy-Green operators [2].

Denote by Δ the triangle $\Delta = \{z \in \mathbb{C} : 0 < \text{Im } z < 1 - |\text{Re } z|\}$. Note that $\text{Area}(\Delta) = 1$.

Theorem 3.2 *Let $\Sigma = \Sigma_\Delta$. Let $A(Z) : \mathbb{H}_0 \rightarrow \mathbb{H}_0$, be a family of bounded linear operators continuously depending on $Z \in \mathbb{H}_0$ and such that $A(Z) = 0$ for $Z \in \mathbb{H}_0 \setminus \Sigma$. Suppose that $A(Z) : \mathbb{H}_s \rightarrow \mathbb{H}_s$ is bounded for every $s \in [0, s_0]$. Furthermore, suppose there is $a < 1$ so that for all $Z \in \mathbb{H}_0$*

$$\|A(Z)\|_{\mathbb{H}_s} \leq a. \quad (9)$$

Then there exists $p > 2$ such that for every point $(z^0, w^0) \in \Sigma$ there is a solution $Z \in W^{1,p}(\mathbb{D}, \mathbb{H}_0)$ of (2) such that $Z(\overline{\mathbb{D}}) \subset \overline{\Sigma}$, $(z^0, w^0) \in Z(\mathbb{D})$, $\text{Area}(Z) = 1$, and

$$Z(b\mathbb{D}) \subset b\Sigma. \quad (10)$$

In reducing Theorem 3.1 to Theorem 3.2, we essentially follow Gromov's [14] argument.

Proof of Theorem 3.1. A diffeomorphism whose z -component is an area-preserving map and whose w -components are the identity maps, preserves the form ω . This observation reduces the proof to the case where G is contained in the triangular cylinder $\Sigma := \{(z, w) : z \in \sqrt{\pi}R\Delta\}$.

Put $\tilde{J} = \Phi_*(J_{st})$. Put $P = \Phi_Z$ and $Q = \Phi_{\overline{Z}}$. By Lemma 2.4 the complex representation \tilde{A} of \tilde{J} has the form (6). Furthermore, by Proposition 2.5 (b), $\tilde{A}(Z)$ satisfies (9) for all $Z \in G$ and $s \in [0, s_0]$.

Fix $\varepsilon > 0$. Let $0 \leq \chi \leq 1$ be a smooth cut-off function with support in G and such that $\chi = 1$ on $\Phi((r - \varepsilon)\overline{\mathbb{B}}^\infty)$. Define $A = \chi\tilde{A}$. Since $\chi \leq 1$, the estimate (9) holds for A .

Let $p = \Phi(0)$. By Theorem 3.2 there exists a solution Z of (2) such that $p \in Z(\mathbb{D})$, $Z(b\mathbb{D}) \subset b\Sigma$ and $\text{Area}(Z) = \pi R^2$. Denote by $D \subset \mathbb{D}$ a connected component of the pre-image $Z^{-1}(\Phi((r - \varepsilon)\mathbb{B}^\infty))$. Then $X = \Phi^{-1}(Z(D))$ is a closed J_{st} -complex curve in $(r - \varepsilon)\mathbb{B}^\infty$ with boundary contained in $(r - \varepsilon)b\mathbb{B}^\infty$. Furthermore, $0 \in X$ and $\text{Area}(X) \leq \pi R^2$.

Consider the canonical projection $\pi_n : \mathbb{H} \rightarrow \mathbb{C}^n$, $\pi_n : z = (z_1, z_2, \dots) \mapsto (z_1, z_2, \dots, z_n)$. Set $Z' = \Phi^{-1} \circ Z$. Since Z' is a Hilbert space valued holomorphic function in a neighborhood of

\overline{D} , by means of the Cauchy integral, the sequence $\pi_n \circ Z'$ uniformly converges to Z' on \overline{D} as $n \rightarrow \infty$.

Fix n big enough such that $(\sum_{j=1}^n |Z'_j(\zeta)|^2)^{1/2} > (1 - 2\varepsilon)r$ for every $\zeta \in bD$. Then $X_n := (\pi_n \circ \Phi^{-1} \circ Z)(D) \cap (r - 2\varepsilon)\mathbb{B}^n$ is a closed complex (with respect to J_{st}) curve through the origin in \mathbb{B}^n . By the classical result due to Lelong (see, e.g., [9]) we have $\text{Area}(X_n) \geq \pi(r - 2\varepsilon)^2$. Since $\text{Area}(X_n) \leq \text{Area}(X)$ and ε is arbitrary, we obtain $r \leq R$ as desired. ■

Corollary 3.3 *Let $r, R > 0$ and let G be an open subset in $\Sigma_{R\mathbb{D}}$. Suppose that there exists a symplectomorphism $\Phi : r\mathbb{B}^\infty \rightarrow G$ of class C^1 with respect to the \mathbb{H}_0 -norm. Let $s_0 > 0$. Suppose that the tangent maps $d\Phi(Z) : \mathbb{H}_0 \rightarrow \mathbb{H}_0$ as well as their inverses $(d\Phi(Z))^{-1}$ are bounded uniformly in $Z \in r\mathbb{B}^\infty$. Furthermore, suppose that the antiholomorphic derivative $\Phi_{\overline{Z}}(Z)$ as an operator $\mathbb{H}_0 \rightarrow \mathbb{H}_{s_0}$ is bounded uniformly in $Z \in r\mathbb{B}^\infty$. Then $r \leq R$.*

The proof goes along the same lines as the proof of Theorem 3.1 because the hypotheses imply that \hat{A} satisfies (9).

Corollary 3.3 can be applied, for example, to symplectomorphisms of the form $\Phi = h + K$, where h is a holomorphic transformation of \mathbb{H}_0 and $K : \mathbb{H}_0 \rightarrow \mathbb{H}_{s_0}$, i.e., K is a compact map $\mathbb{H}_0 \rightarrow \mathbb{H}_0$. Hence, we obtain a generalization of Kuksin's result, in which h is linear. One can view Corollary 3.3 as an infinitesimal version of theorem of Kuksin.

If (\mathbb{H}_s) is the Sobolev scale, the main assumption of Corollary 3.3 means that the antiholomorphic derivative $\Phi_{\overline{Z}}(Z)$ is a smoothing operator. This condition is more restrictive than the assumptions of Theorem 3.1 which do not require any smoothing property. In the last section, we apply Theorem 3.1 to the discrete Schrödinger equation. The flow of the latter does not have a smoothing effect.

4 Cauchy integral for vector functions

We recall the modifications of the Cauchy-Green operator considered in [21]. Their properties are well-known in the scalar case [3, 18, 23]. The difference is that we need them for Hilbert space-valued functions.

4.1 Cauchy integral and related operators

Consider the arcs $\gamma_1 = \{e^{i\theta} : 0 < \theta < \pi/2\}$, $\gamma_2 = \{e^{i\theta} : \pi/2 < \theta < \pi\}$, $\gamma_3 = \{e^{i\theta} : \pi < \theta < 2\pi\}$ on the unit circle in \mathbb{C} . Define the functions

$$R(\zeta) = e^{3\pi i/4}(\zeta - 1)^{1/4}(\zeta + 1)^{1/4}(\zeta - i)^{1/2}, \quad X(\zeta) = R(\zeta)/\sqrt{\zeta}.$$

Here we choose the branch of R continuous in $\overline{\mathbb{D}}$ satisfying $R(0) = e^{3\pi i/4}$. For definiteness, we also choose the branch of $\sqrt{\zeta}$ continuous in \mathbb{C} with deleted positive real line, $\sqrt{-1} = i$.

Then $\arg X$ on arcs γ_j , $j = 1, 2, 3$ is equal to $3\pi/4$, $\pi/4$ and 0 respectively. Therefore, the function X satisfies the boundary conditions

$$\begin{cases} \operatorname{Im}((1+i)X(\zeta)) = 0, & \zeta \in \gamma_1, \\ \operatorname{Im}((1-i)X(\zeta)) = 0, & \zeta \in \gamma_2, \\ \operatorname{Im} X(\zeta) = 0, & \zeta \in \gamma_3, \end{cases} \quad (11)$$

which represent the lines through 0 parallel to the sides of the triangle Δ .

Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be a measurable function. The Cauchy (or Cauchy-Green) operator is defined by

$$Tf(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f(t)dt \wedge d\bar{t}}{t - \zeta}.$$

The operator $T : L^p(\mathbb{D}) \rightarrow W^{1,p}(\mathbb{D})$ is bounded for $p > 1$, and $(\partial/\partial\bar{\zeta})Tf = f$ as Sobolev's derivative, i.e., T solves the $\bar{\partial}$ -problem in \mathbb{D} . Furthermore, Tf is holomorphic on $\mathbb{C} \setminus \overline{\mathbb{D}}$.

Let Q be a function in \mathbb{D} . Introduce the modified Cauchy-Green operator

$$T_Q f(\zeta) = Q(\zeta) \left(T(f/Q)(\zeta) + \zeta^{-1} \overline{T(f/Q)(1/\bar{\zeta})} \right).$$

It can be viewed as a symmetrization of the operator T with the weight Q . We will need only the operators corresponding to two special weights, namely

$$T_1 f = T_Q f + 2i \operatorname{Im} Tf(1) \quad \text{with } Q = \zeta - 1$$

and

$$T_2 f = T_Q f \quad \text{with } Q = R.$$

Note that

$$T_1 f(\zeta) = Tf(\zeta) - \overline{Tf(1/\bar{\zeta})}. \quad (12)$$

We also define formal derivatives

$$S_j f(\zeta) = \frac{\partial}{\partial \zeta} T_j f(\zeta)$$

as integrals in the sense of the Cauchy principal value. We recall the following facts [21].

Proposition 4.1 *The operators T_j, S_j ($j = 1, 2$) enjoy the following properties:*

- (i) *Each $S_j : L^p(\mathbb{D}) \rightarrow L^p(\mathbb{D})$ is a bounded linear operator for $p_1 < p < p_2$. Here for S_1 one has $p_1 = 1$ and $p_2 = \infty$ and for S_2 one has $p_1 = 4/3$ and $p_2 = 8/3$. Moreover, for $f \in L^p(\mathbb{D})$, one has $S_j f(\zeta) = (\partial/\partial\zeta)T_j f(\zeta)$ as Sobolev's derivative.*
- (ii) *Each $T_j : L^p(\mathbb{D}) \rightarrow W^{1,p}(\mathbb{D})$ is a bounded linear operator for $p_1 < p < p_2$. Moreover, for $f \in L^p(\mathbb{D})$, one has $(\partial/\partial\bar{\zeta})T_j f = f$ on \mathbb{D} as Sobolev's derivative.*

- (iii) For every $f \in L^p(\mathbb{D})$, $2 < p < p_2$, the function $T_1 f$ satisfies $\operatorname{Re} T_1 f|_{\partial\mathbb{D}} = 0$ whereas $T_2 f$ satisfies the same boundary conditions (11) as X .
- (iv) Each $S_j : L^2(\mathbb{D}) \rightarrow L^2(\mathbb{D})$, $j = 1, 2$, is an isometry.
- (v) The function $p \mapsto \|S_j\|_{L^p}$ approaches $\|S_j\|_{L^2} = 1$ as $p \searrow 2$.

We need to extend Proposition 4.1 to Hilbert space-valued functions.

4.2 Operators on spaces of vector functions

For definiteness we only consider functions $\mathbb{D} \rightarrow \mathbb{H}$, where as usual \mathbb{H} is a separable Hilbert space. A function $u : \mathbb{D} \rightarrow \mathbb{H}$ is called *simple* if it takes only a finite number of values h_j , $j = 1, \dots, m$ and every preimage $u^{-1}(h_j)$ is a measurable set. The function u is called *strongly measurable* if there exists a sequence of simple functions (u_n) converging to u in the norm of \mathbb{H} . A vector function is called *weakly measurable* if for every $h \in \mathbb{H}$, the function $\zeta \mapsto \langle u(\zeta), h \rangle$ is measurable. By Pettis's theorem [24] for functions with values in separable spaces these two notions coincide, so we will use the term *measurable*. Note that simple functions are dense in $L^p(\mathbb{D}, \mathbb{H})$.

Let $P : L^p(\mathbb{D}) \rightarrow L^p(\mathbb{D})$ be a bounded linear operator. We say that P *extends to* $L^p(\mathbb{D}, \mathbb{H})$ if there is a unique bounded linear operator $P_{\mathbb{H}} : L^p(\mathbb{D}, \mathbb{H}) \rightarrow L^p(\mathbb{D}, \mathbb{H})$ such that for every $u \in L^p(\mathbb{D})$ and $h \in \mathbb{H}$ we have $P_{\mathbb{H}}(uh) = P(u)h$. We will usually omit the index \mathbb{H} in $P_{\mathbb{H}}$.

Proposition 4.2 (i) Every bounded linear operator $P : L^p(\mathbb{D}) \rightarrow L^p(\mathbb{D})$ extends to $L^p(\mathbb{D}, \mathbb{H})$, $1 \leq p < \infty$, and the extension has the same norm as P .

(ii) For $p > 2$ the operators T, T_1 are bounded linear operators $L^p(\mathbb{D}, \mathbb{H}) \rightarrow C^\alpha(\mathbb{D}, \mathbb{H})$ with $\alpha = (p - 2)/p$.

(iii) For $u \in L^p(\mathbb{D}, \mathbb{H})$ for the same p as in Proposition 4.1, we have

$$\frac{\partial T u}{\partial \bar{\zeta}} = u, \quad \frac{\partial T_j u}{\partial \bar{\zeta}} = u, \quad \frac{\partial T u}{\partial \zeta} = S u, \quad \frac{\partial T_j u}{\partial \zeta} = S_j u, \quad j = 1, 2$$

as weak derivatives.

(iv) The operators T, T_1, T_2 are bounded linear operators $L^p(\mathbb{D}, \mathbb{H}) \rightarrow W^{1,p}(\mathbb{D}, \mathbb{H})$ for the same p as in Proposition 4.1.

Proof. (i) If P is a singular integral operator, the result follows because \mathbb{H} is a UMD space [8]. For a general bounded linear operator the result follows because \mathbb{H} is so called p -space [15], which means exactly the same as Proposition 4.2 (i). For completeness we give a direct proof of Proposition 4.2 (i) in Appendix. Since the operators T, T_1, T_2, S, S_1 , and S_2 are bounded linear operators in $L^p(\mathbb{D})$ for appropriate $p > 1$, they extend to $L^p(\mathbb{D}, \mathbb{H})$.

(ii) Let $u = \sum_{j=1}^n \chi_k h_k : \mathbb{D} \rightarrow \mathbb{H}$ be a simple function. Here $h_k \in \mathbb{H}$ and χ_k is a characteristic function of a measurable set in \mathbb{D} . Then $T\chi_k$ are defined and Tu can be defined by $Tu = \sum_{k=1}^n (T\chi_k)h_k$. The range of this extended operator T is contained in the finite dimensional space $\mathbb{H}_n := \text{Span}\{h_k : 1 \leq k \leq n\} \subset \mathbb{H}$. The proof of the corresponding result for scalar functions (see for instance [23], Theorem 1.19) goes through with no changes for the operator T extended to \mathbb{H}_n -valued functions. Since simple functions are dense in $L^p(\mathbb{D}, \mathbb{H})$, the proposition follows for T . For the operator T_1 the desired result follows by the formula (12).

(iii) The bounded linear form $L^1(\mathbb{D}) \rightarrow \mathbb{C}$, $u \mapsto \int_{\mathbb{D}} u(\zeta) d^2\zeta$ has the norm equal to 1. By (i) it extends to $L^1(\mathbb{D}, \mathbb{H})$ and the extension has the norm 1. This definition of the integral for \mathbb{H} -valued functions is equivalent to Bochner's integral [24]. The result (iii) follows because it holds for scalar-valued functions and because all the operators in question are bounded in $L^p(\mathbb{D}, \mathbb{H})$. For example, we prove that $\partial Tu / \partial \zeta = Su$. We need to show that for every (scalar-valued) test function ϕ

$$\int_{\mathbb{D}} ((Su)\phi + (Tu)\phi) d^2\zeta = 0. \quad (13)$$

Fix ϕ . Since the operators S and T are bounded in $L^p(\mathbb{D}, \mathbb{H})$, the left-hand part of (13) defines a bounded linear form (in u) on $L^p(\mathbb{D}, \mathbb{H})$. Every simple function u satisfies (13) because the result holds for scalar-valued functions. The conclusion now follows by density of simple functions in $L^p(\mathbb{D}, \mathbb{H})$.

(iv) is immediate by (i) and (iii). ■

Although the result of (ii) for T_2 still holds, the argument does not go through directly. We do not need this result here.

We need a version of Proposition 4.2 (i) for two operators acting on different components of a vector-valued function.

Proposition 4.3 *Let X_1 and X_2 be Hilbert spaces and let $X = X_1 \oplus X_2$ be their Hilbert direct sum. Let $P_j : L^p(\mathbb{D}) \rightarrow L^p(\mathbb{D})$ be bounded linear operators, $p \geq 2$, $j = 1, 2$. Let $P = (P_1, P_2) : L^p(\mathbb{D}, X) \rightarrow L^p(\mathbb{D}, X)$, here P_j stands for its extension to $L^p(\mathbb{D}, X_j)$. Then $\|P\|_{L^p(\mathbb{D}, X)} \leq C_p m_p$, here $C_p = 2^{\frac{1}{2} - \frac{1}{p}}$, and $m_p = \max_{j=1,2} \|P_j\|_{L^p(\mathbb{D})}$.*

Proof. For $x = (x_1, x_2) \in X$ we introduce the p -norm

$$\|x\|_p = (\|x_1\|^p + \|x_2\|^p)^{1/p}.$$

This norm is equivalent to the Hilbert space norm $\|x\| = \|x\|_2$ on X . Furthermore, for $p \geq 2$,

$$\|x\|_p \leq \|x\|_2 \leq C_p \|x\|_p.$$

Here $C_p = \sup\{\|x\|_p^{-1} : x \in X, \|x\|_2 = 1\}$. Then $C_p = 2^{\frac{1}{2} - \frac{1}{p}}$ because the maximum is attained when $\|x_1\| = \|x_2\|$.

The space X equipped with the p -norm is a Banach space, we denote it by $X^{(p)}$. For $p \geq 2$ we immediately obtain

$$\|f\|_{L^p(\mathbb{D}, X^{(p)})} \leq \|f\|_{L^p(\mathbb{D}, X)} \leq C_p \|f\|_{L^p(\mathbb{D}, X^{(p)})}.$$

For $f = (f_1, f_2) \in L^p(\mathbb{D}, X)$ we have

$$\begin{aligned} \|Pf\|_{L^p(\mathbb{D}, X)}^p &\leq C_p^p \|Pf\|_{L^p(\mathbb{D}, X^{(p)})}^p = C_p^p \int_{\mathbb{D}} (\|P_1 f_1(\zeta)\|_{X_1}^p + \|P_2 f_2(\zeta)\|_{X_2}^p) d^2 \zeta \\ &\leq C_p^p m_p^p \int_{\mathbb{D}} (\|f_1(\zeta)\|_{X_1}^p + \|f_2(\zeta)\|_{X_2}^p) d^2 \zeta = C_p^p m_p^p \|f\|_{L^p(\mathbb{D}, X^{(p)})}^p \leq C_p^p m_p^p \|f\|_{L^p(\mathbb{D}, X)}^p, \end{aligned}$$

hence the conclusion. ■

5 Proof of Theorem 3.2

Let $\Phi : \mathbb{D} \rightarrow \Delta$ be a biholomorphism satisfying $\Phi(\pm 1) = \pm 1$ and $\Phi(i) = i$. Note that $\Phi \in W^{1,p}(\mathbb{D})$ for $p \geq 2$ close enough to 2 by the Christoffel-Schwarz formula. We look for a solution $Z = (z, w) : \mathbb{D} \rightarrow \mathbb{H}_0$ of (2) of class $W^{1,p}(\mathbb{D}, \mathbb{H}_0)$, $p > 2$, in the form

$$\begin{cases} z = T_2 u + \Phi, \\ w = T_1 v - T_1 v(\tau) + w^0. \end{cases} \quad (14)$$

for some $\tau \in \mathbb{D}$; hence, $w(\tau) = w^0$.

This form ensures that z satisfies the desired boundary conditions, namely, takes $b\mathbb{D}$ to $b\Delta$. We do not have specific boundary conditions on the w -components; in (14) each component w_j takes $b\mathbb{D}$ to a line $\operatorname{Re} w_j = \text{const.}$ This way the w -components will not contribute to the area of the disc. We could use the operator T_2 in both lines of (14), however, the choice we make here seems more natural.

The Cauchy-Riemann equation (2) for Z of the form (14) turns into the integral equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = A(z, w) \begin{pmatrix} \overline{S_2 u} + \overline{\Phi'} \\ \overline{S_1 v} \end{pmatrix}. \quad (15)$$

We will show that there exists a solution of (14, 15) so that $z(\tau) = z^0$ for some $\tau \in \mathbb{D}$.

We first analyze the equation (15) for fixed $Z \in C(\overline{\mathbb{D}}, \mathbb{H}_0)$. For every $\zeta \in \mathbb{D}$, the operator $A(Z(\zeta))$ is bounded in \mathbb{H}_s and satisfies $\|A(Z(\zeta))\|_{\mathbb{H}_s} \leq a < 1$, $s \in [0, s_0]$. Then $A(Z)$ defines an operator on $L^p(\mathbb{D}, \mathbb{H}_s)$ such that $\|A(Z)\|_{L^p(\mathbb{D}, \mathbb{H}_s)} \leq a$.

Let $m_p = \max_{j=1,2} \|S_j\|_{L^p(\mathbb{D})}$. Then by Proposition 4.3, we have $\|(S_2, S_1)\|_{L^p(\mathbb{D}, \mathbb{H}_s)} \leq C_p m_p$. Since $m_p \rightarrow 1$ and $C_p \rightarrow 1$ as $p \searrow 2$, we can fix $p > 2$ close to 2 such that $a C_p m_p < 1$.

Then for every fixed $Z \in C(\overline{\mathbb{D}}, \mathbb{H}_0)$, by the contraction principle there exists a unique solution $U = (u, v) \in L^p(\mathbb{D}, \mathbb{H}_s)$ of the equation (15) satisfying

$$\|U\|_{L^p(\mathbb{D}, \mathbb{H}_s)} \leq a(C_p m_p \|U\|_{L^p(\mathbb{D}, \mathbb{H}_s)} + \|(\Phi', 0)\|_{L^p(\mathbb{D}, \mathbb{H}_s)}), \quad \|U\|_{L^p(\mathbb{D}, \mathbb{H}_s)} \leq M_1 := \frac{a\|\Phi'\|_{L^p(\mathbb{D})}}{1 - a C_p m_p}.$$

We now obtain an *a priori* estimate for (14, 15). Indeed, by (14) there exists a constant $M > 0$ depending on M_1 and w^0 such that

$$\|Z\|_{C(\overline{\mathbb{D}}, \mathbb{H}_0)} \leq M. \quad (16)$$

We now define a continuous map $\Psi : \mathbb{C} \rightarrow \overline{\mathbb{D}}$

$$\Psi(z) = \begin{cases} \Phi^{-1}(z), & z \in \overline{\Delta}, \\ \Phi^{-1}(b\Delta \cap [z^0, z]), & z \in \mathbb{C} \setminus \overline{\Delta}. \end{cases}$$

Here $[z^0, z]$ is the line segment from z^0 to z , and the intersection $b\Delta \cap [z^0, z]$ consists of a single point.

Let $E = B \times \overline{\mathbb{D}}$, here $B = \{Z \in C(\overline{\mathbb{D}}, \mathbb{H}_0) : \|Z\|_{C(\overline{\mathbb{D}}, \mathbb{H}_0)} \leq M\}$. Introduce the map $F : E \rightarrow E$, $F : (z, w, \tau) \mapsto (\tilde{z}, \tilde{w}, \tilde{\tau})$ defined by

$$\begin{aligned} \tilde{z} &= T_2 u + \Phi, \\ \tilde{w} &= T_1 v - T_1 v(\tau) + w_0, \\ \tilde{\tau} &= \Psi(z^0 - T_2 u(\tau)). \end{aligned}$$

Here (u, v) is a solution of (15). By (16) the map F is well defined.

The map F is continuous. Indeed, let $Z_0 \in C(\overline{\mathbb{D}}, \mathbb{H}_0)$. Then $Z_0(\overline{\mathbb{D}}) \subset \mathbb{H}_0$ is compact. Then if $Z \in C(\overline{\mathbb{D}}, \mathbb{H}_0)$ is close to Z_0 , then $A(Z(\zeta))$ and $A(Z_0(\zeta))$ as operators on \mathbb{H}_0 are close in the operator norm uniformly in $\zeta \in \overline{\mathbb{D}}$. Then the corresponding solutions U and U_0 of (15) are close in $L^p(\mathbb{D}, \mathbb{H}_0)$, and the continuity of F follows.

By Proposition 4.2, $F(E) \subset W^{1,p}(\mathbb{D}, \mathbb{H}_s)$. By Proposition 2.7 the inclusion $W^{1,p}(\mathbb{D}, \mathbb{H}_s) \subset C(\overline{\mathbb{D}}, \mathbb{H}_0)$ is compact, hence $F(E)$ is compact. Now by Schauder's principle the continuous compact map F on a convex set E has a fixed point (z, w, τ) . The fixed point satisfies (14), (15) and $\tau = \Psi(z^0 - T_2 u(\tau))$.

By (14) and (15), the map $Z = (z, w) \in W^{1,p}(\mathbb{D}, \mathbb{H}_0)$, satisfies the Cauchy-Riemann equations (2), and $w(\tau) = w^0$.

We state the rest of the conclusions of Theorem 3.2 in the following

Lemma 5.1 (i) $\tau \in \mathbb{D}$ and $z(\tau) = z^0$.

(ii) The map z satisfies $z(\overline{\mathbb{D}}) \subset \overline{\Delta}$, $z(b\mathbb{D}) \subset b\Delta$, and $\deg z = 1$; here $\deg z$ denotes the degree of the map $z|_{b\mathbb{D}} : b\mathbb{D} \rightarrow b\Delta$. In particular, Z satisfies (10).

(iii) $\text{Area}(Z) = 1$.

Proof. We closely follow [21]. If $\tau \in b\mathbb{D}$, then $z^0 - T_2 u(\tau) \notin \Delta$, and the line connecting z^0 and $z^0 - T_2 u(\tau)$ intersects a certain edge of Δ . But they can not intersect because by the boundary conditions on $T_2 u$, they are parallel. Now that $\tau \in \mathbb{D}$, by the definition of Ψ , we have $\Phi(\tau) = z^0 - T_2 u(\tau)$, and $z(\tau) = z^0$.

Since $A = 0$ on $\mathbb{H}_0 \setminus \Sigma$, the function z is holomorphic at every $\zeta \in \mathbb{D}$ for which $z(\zeta) \notin \overline{\Delta}$. Then by the maximum principle, $z(\overline{\mathbb{D}}) \subset \overline{\Delta}$. By the boundary conditions on T_2u , the map z takes the arcs γ_j of $b\mathbb{D}$ to the corresponding edges of Δ , hence $\deg z = 1$.

By the structure of the standard symplectic form, all components separately contribute to $\text{Area}(Z)$. Since each w_j -component of Z takes $b\mathbb{D}$ to a real line, it does not contribute to the area. Since $z(b\mathbb{D}) = b\Delta$ and $\deg z = 1$, we have $\text{Area}(Z) = \text{Area}(\Delta) = 1$ as desired. We refer to [21] for more details. ■

The proof of Theorem 3.2 is complete.

6 Application to the discrete Schrödinger equation

Consider the following system of equations

$$iu'_n + f(|u_n|^2)u_n + \sum_k a_{nk}u_k = 0. \quad (17)$$

Here $u(t) = (u_n(t))_{n \in \mathbb{Z}}$, $u_n(t) \in \mathbb{C}$, $t \geq 0$. We use the notation $u'_n = du_n/dt$. We assume that $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and its derivative are continuous on the positive reals, furthermore, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} [xf'(x)] = 0$. For example, one can take $f(x) = x^p$ with real $p > 0$. The hypotheses on the function f are imposed in order for the flow of (17) to be C^1 smooth. We suppose that $A = (a_{nk})$ is an infinite matrix independent of t . Furthermore, A is a hermitian matrix, that is, $a_{nk} = \overline{a_{kn}}$. For simplicity we also assume that the entries a_{nk} are uniformly bounded and there exists $m > 0$ such that $a_{nk} = 0$ if $|n - k| > m$.

The equation (17) with $f(x) = x$ is called the discrete self-trapping equation [11]. The special case with $a_{nk} = 1$ if $|n - k| = 1$ and $a_{nk} = 0$ otherwise, is the discrete nonlinear (cubic) Schrödinger equation:

$$iu'_n + |u_n|^2u_n + u_{n-1} + u_{n+1} = 0.$$

There are other discretizations of the Schrödinger equation, in particular, the Ablowitz-Ladik model

$$iu'_n + (1 + |u_n|^2)(u_{n-1} + u_{n+1}) = 0.$$

The latter does not have the form (17), but it can be treated in a similar way. In the special case $A = 0$, the equation (17) can be solved explicitly:

$$u_n(t) = e^{itf(|u_n(0)|^2)}u_n(0).$$

The equation (17) in this special form was suggested to us by Stephan de Bievre.

The equation (17) can be written in the Hamiltonian form:

$$u'_n = i \frac{\partial H}{\partial \overline{u_n}}.$$

The Hamiltonian H is given by

$$H = \sum_n F(|u_n|^2) + \sum_{n,k} a_{nk} \overline{u_n} u_k,$$

here $F' = f$ and $F(0) = 0$. The equation (17) preserves the $l^2(\mathbb{Z})$ norm $\|u\|_{l^2} = (\sum_n |u_n|^2)^{1/2}$. Hence, the flow $u(0) \mapsto u(t)$ of (17) is globally defined on $l^2(\mathbb{Z})$ and preserves the standard symplectic form $\omega = (i/2) \sum_n du_n \wedge d\overline{u_n}$.

We claim that Theorem 3.1 applies to (17), hence, the non-squeezing property holds for the flow of (17). Of course, the ball in Theorem 3.1 need not have a center at the origin.

Consider the standard Hilbert scale \mathbb{H}_s , $s \in \mathbb{R}$, defined in Section 2.3 using $\mathbb{H}_0 = l^2(\mathbb{Z})$ and $\theta_n = (1 + n^2)^{1/2}$. Namely, $\mathbb{H}_s = \{u = (u_n)_{n \in \mathbb{Z}} \mid \|u\|_s < \infty\}$, where $\|u\|_s^2 = \sum_n |u_n|^2 \theta_n^{2s}$ is the norm in \mathbb{H}_s .

Let $u(t)$ be a solution of (17) such that $\|u(t)\|_0 = \|u(0)\|_0 \leq M$. The derivative of the flow of (17) at $u(t)$ is the flow of the linear equation

$$iv'_n + a_n v_n + b_n \overline{v_n} + \sum_k a_{nk} v_k = 0.$$

Here

$$a_n = f'(|u_n|^2)|u_n|^2 + f(|u_n|^2), \quad b_n = f'(|u_n|^2)u_n^2.$$

We claim that the operator $v(0) \mapsto v(t)$ is bounded in \mathbb{H}_s for all $s \in \mathbb{R}$. We have

$$(|v_n|^2)' = 2\operatorname{Re}(v'_n \overline{v_n}) = -2\operatorname{Im}(a_n |v_n|^2 + b_n \overline{v_n}^2 + \sum_k a_{nk} \overline{v_n} v_k).$$

Using the assumptions on f , u , and A , we obtain the estimate

$$|(|v_n|^2)'| \leq C_1(|v_n|^2 + |v_n| \sum_{|k-n| \leq m} |v_k|),$$

with C_1 depending on f , A , and M . Note that $(2/5)^{1/2} \leq \theta_n/\theta_{n+1} \leq (5/2)^{1/2}$ for all $n \in \mathbb{Z}$. Then we have

$$(|v_n|^2 \theta_n^{2s})' \leq C_2(|v_n|^2 \theta_n^{2s} + |v_n| \theta_n^s \sum_{|k-n| \leq m} |v_k| \theta_k^s).$$

Here C_2 depends on C_1 and s . Since $2(|v_n| \theta_n^s)(|v_k| \theta_k^s) \leq |v_n|^2 \theta_n^{2s} + |v_k|^2 \theta_k^{2s}$, summation by n yields

$$\sum (|v_n|^2 \theta_n^{2s})' \leq C_3 \sum |v_n|^2 \theta_n^{2s}.$$

Here $C_3 = (2m + 2)C_2$. Thus,

$$(\|v(t)\|_s^2)' \leq C_3 \|v(t)\|_s^2.$$

By the Grönwall lemma

$$\|v(t)\|_s^2 \leq e^{C_3 t} \|v(0)\|_s^2.$$

Hence, the linear operator $v(0) \mapsto v(t)$ is bounded in \mathbb{H}_s , $s \in \mathbb{R}$ by a constant $C = e^{C_3 t/2}$, depending on t , s , $\|u\|_0$ and bounds on f and A . This completes the proof of the claim.

7 Appendix: Proof of Proposition 4.2 (i)

We need the following

Lemma 7.1 *For every $p_0 > 1$ there is a map $\mathbb{H} \rightarrow L^{p_0}(0, 1)$ which is an isometric embedding $\mathbb{H} \rightarrow L^p(0, 1)$ for all $1 \leq p \leq p_0$.*

Proof. Without loss of generality assume that $p_0 = 2m$ is an even integer. Choose $A > m$. Let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis of \mathbb{H} . For every $c = \sum_{n=1}^\infty c_n e_n$ define a “lacunary” Fourier series

$$f(t) = \sum_{n=1}^\infty c_n e^{2\pi i A^n t}.$$

It turns out (see [13], proof of Theorem 3.7.4 with $r = 1$) that $\|f\|_{2m} = \|f\|_2$. Since the function $p \mapsto \|f\|_p$ is nondecreasing and logarithmically convex, we conclude that $\|f\|_p = \|f\|_2 = \|c\|_{\mathbb{H}}$ for all $1 \leq p \leq p_0$. Hence $c \mapsto f$ is an isometry. ■

Identify \mathbb{H} with a closed subspace of $L^p(0, 1)$. Then the value $u(\zeta)$ of $u \in L^p(\mathbb{D}, \mathbb{H})$ is a function $u(\zeta) \in L^p(0, 1)$. Hence u defines the map $\mathbb{D} \times (0, 1) \rightarrow \mathbb{C}$, $(\zeta, t) \mapsto u(\zeta)(t)$ which we denote by the same letter u . Consider a simple function $u \in L^p(\mathbb{D}, \mathbb{H})$. Then $u(\zeta, t) = u(\zeta)(t) = \sum_{k=1}^n \chi_k(\zeta) h_k(t)$. Here χ_k is the characteristic function of a measurable set $E_k \subset \mathbb{D}$, the sets E_k are disjoint and $h_k(t) \in \mathbb{H} \subset L^p(0, 1)$. Define

$$(P_{\mathbb{H}}u)(\zeta, t) = \sum_{k=1}^n (P\chi_k)(\zeta) h_k(t). \quad (18)$$

Obviously both u and $P_{\mathbb{H}}u$ are measurable as maps $\mathbb{D} \times (0, 1) \rightarrow \mathbb{C}$. Then by Fubini’s theorem

$$\begin{aligned} \|P_{\mathbb{H}}u\|_p^p &= \int_{\mathbb{D}} \int_0^1 |(P_{\mathbb{H}}u)(\zeta, t)|^p dt d^2\zeta = \int_0^1 \int_{\mathbb{D}} |(P_{\mathbb{H}}u)(\zeta, t)|^p d^2\zeta dt \\ &\leq \|P\|_p^p \int_0^1 \int_{\mathbb{D}} |u(\zeta, t)|^p d^2\zeta dt = \|P\|_p^p \|u\|_p^p. \end{aligned}$$

Since simple functions are dense in $L^p(\mathbb{D}, \mathbb{H})$, the operator $P_{\mathbb{H}}$ uniquely extends to all $L^p(\mathbb{D}, \mathbb{H})$, and $\|P_{\mathbb{H}}\|_p = \|P\|_p$. Note that for $u \in L^p(\mathbb{D}, \mathbb{H})$ the image $P_{\mathbb{H}}u \in L^p(\mathbb{D}, \mathbb{H})$ because it is true for simple functions. Furthermore, if $u = \varphi h$, $\varphi \in L^p(\mathbb{D})$, $h \in \mathbb{H}$, then $P_{\mathbb{H}}u = (P\varphi)h$. Indeed, let $\varphi_n : \mathbb{D} \rightarrow \mathbb{C}$ be a sequence of simple functions converging to φ in $L^p(\mathbb{D})$. Then $\varphi_n h \rightarrow \varphi h$ in $L^p(\mathbb{D}, \mathbb{H})$ and

$$P_{\mathbb{H}}u = P_{\mathbb{H}}(\varphi h) = \lim_{n \rightarrow \infty} P_{\mathbb{H}}(\varphi_n h) = h \lim_{n \rightarrow \infty} (P\varphi_n) = (P\varphi)h.$$

Proposition 4.2 (i) is proved.

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